Minimal Cut Set List Quantification with Mutually Exclusive Events and High Probabilities – An Improvement to the MCS BDD Split Approach

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Abstract:
Seismic analyses within a Probabilistic Safety Assessment (PSA) differ from typical internal event analyses in several important ways. One key distinction arises from the nature of strong seismic events. Earthquakes with high peak ground acceleration occur infrequently, but they are likely to cause the simultaneous unavailability of many components. As a result, the minimal cut set list associated with such events is characterized by a low initiating event frequency and a high conditional core damage probability (CCDP). A conservative approach sets the CCDP to one, effectively equating the result to the initiating event frequency. However, this can be overly pessimistic and prevents meaningful comparison of the relative importance of safety measures for high-intensity seismic scenarios.

A more realistic estimate can be obtained by quantifying the minimal cut set list using a Binary Decision Diagram (BDD)–based method. The algorithm implemented in RiskSpectrum can process large minimal cut set lists through multiple heuristics and approximations. One such heuristic divides the list into two parts: a dominant subset representing nearly the entire top value, which is quantified using the BDD, and a remaining subset with insignificant contribution to the top value, quantified using the Min-Cut Upper Bound method. For internal events, the BDD-treated portion is typically much smaller than the remainder. Seismic analyses, however, challenge this assumption. If we limit the number of cut sets allocated to the BDD, the CCDP of the remaining part often remains high. This issue becomes particularly disadvantageous when both subsets contain mutually exclusive events, requiring their high CCDP contributions to be summed.

We present a new heuristic for the BDD-based quantification algorithm based on a new way to create modules that mitigates this problem and enables more realistic quantification of high-intensity seismic events.

1. INTRODUCTION

Quantification of a Minimal Cut Set (MCS) list is challenged by different factors. In this paper we will examine and propose a solution to manage situations when an MCS list contains a combination of mutually exclusive events (MUX events) and many high probability events – such that the conditional probability of an MCS is high. The described situation can occur in seismic analysis, especially when high peak ground acceleration is analysed [9].

In scenarios with high peak ground acceleration, many high probability events will be present in the MCS list. And these events will not only be present in a few MCSs. Most of the events in the MCS list will have a high probability. This means that the conditional failure probability, given that the initiating event has happened, will be very high for very many cut sets. And the same events can be expected also to be present in many MCSs.

In addition, modern PSAs do not only include events that are represented as independent failures. PSA models also include events that are defined as mutually exclusive. This, in combination with the large number of high probability events, can be really challenging for the accuracy of the MCS list quantification [10]. Standard methods like Rare Event Approximation or Min Cut Upper Bound (MCUB) might significantly over-approximate the actual sequence or consequence frequency.

This paper will examine how and why this can be a challenge even for a quantification algorithm based on Binary Decision Diagrams [5] and will propose an approach that solves the problem.

2. WHY SEISMIC ASSESSMENT IS AN EXAMPLE OF A CHALLENGING SCENARIO

A seismic model is typically extending an existing PSA model with additional failure modes, fragilities, representing the probability that the component(s) will fail due to the earthquake. The earthquake, the seismic hazard, is analysed from events with a low magnitude (often represented by Peak Ground Acceleration, PGA) to high magnitude events. The hazard is in reality a continuum, but to simplify the assessment it is mostly subdivided into intervals. With increasing PGA the frequency of occurrence within that analysed interval decreases. The likelihood of failure of the component(s), the fragility, is increasing. In the last intervals, the failure probability of the components normally approaches 1.0. The last interval is left open ended – which means that components will be considered failing above that PGA threshold (and the core damage frequency therefore will equal the hazard frequency). This threshold obviously must be set so that such interval is not dominating the results.

This also means that the intervals just before the last interval may have many components with a high failure probability, given the seismic event. Since the fragilities are spread widely in the PSA model, there may be a lot of combinations of such events. The conditional failure probability given that the seismic hazard has occurred can be expected to be high (close to 1.0 – but not equal to 1.0). Such intervals should also not typically be expected to dominate the results from the seismic risk assessment. To use MCUB (which clearly will be conservative due to the amount of high probability events) in such analysis is therefore not normally an issue. However, it is desirable that the assessment can demonstrate that the conditional probability is not 1.0.

3. AN ALGORITHM FOR SCALABLE BDD ON AN MCS LIST

An alternative quantification approach for minimal cut set lists that in theory offers higher precision than first order approximations is based on Binary Decision Diagrams (BDDs) [6, 7, 8].

3.1. Overview of the MCS BDD approach

The process implemented in the RiskSpectrum MCS BDD has been described previously in [1, 2, 3, 4]. This paper describes management of a special case within the algorithm. A brief introduction to the method is presented below.

The MCS BDD algorithm uses a pivotal decomposition method to construct the BDD structure from the MCS list. By continually picking up pivotal elements (decision variables) from the MCS list, both failure and success branches are being further developed. When all the branches have reached "1" (failure state) or "0" (success state), the BDD structure is generated completely. An important process of building the BDD structure is to select the pivotal element. The pivotal element is selected from the remaining MCS list and appended to the branch of the current pivotal node under construction. The order of pivotal elements is defined dynamically by an algorithm that is based on number of replications of the event and the cut set order.

Each node that is added is either included as an Exact node, or as an Approximate node. The use of Exact or Approximate nodes is an efficient way of ensuring scalability of the MCS BDD method. The concept of Exact and Approximate is not the primary objective of this paper, and the reader is referred to [2] for more details.

Another important part of the scalability is that the MCS list is split into one part that is treated by BDD, and one part that is treated by MCUB (Min Cut Upper Bound) illustrated in Figure 1. The split of the MCS list into the two parts is defined by an MCS threshold, given in percent of the top value. The top value is first calculated using MCUB for the complete MCS list, to get a top estimate. Thereafter the split can be defined. Let us denote by Q_{BDD} the (part of the) top value calculated by BDD and by Q_{MCUB} the (part of the) top value calculated by MCUB.

Figure 1: Representation of how the MCS list is split into one part calculated using BDD (above MCS no. B) and MCUB part below no. B

<table>
<tr><th></th><th></th><th>Cutset Order</th></tr>
<tr><td rowspan="14">MCUB/Sum, Whole MCS list</td><td rowspan="7">M = BDD quant</td><td>1</td></tr>
<tr><td>2</td></tr>
<tr><td>3</td></tr>
<tr><td>...</td></tr>
<tr><td>B-2</td></tr>
<tr><td>B-1</td></tr>
<tr><td>B</td></tr>
<tr><td rowspan="7">P = MCUB quant</td><td>B+1</td></tr>
<tr><td>B+2</td></tr>
<tr><td>B+3</td></tr>
<tr><td>...</td></tr>
<tr><td>N-2</td></tr>
<tr><td>N-1</td></tr>
<tr><td>N</td></tr>
</table>

Another factor of importance is that BDD may require a lot of memory storage. As most calculations are typically not done for one calculation at a time, the challenge of memory storage increases further. To manage this challenge, there is a setting in which the user can define how many nodes that may be used during the generation of the BDD for one MCS list. As there needs to be consistency in the generation of the results, independent of where the calculation has been performed (RAM memory on the computer – or different computers) as well as if the calculation has been done individually or if it was generated in a run with parallel calculations, the memory consumption has to be managed. When a calculation has reached the amount of BDD nodes that can be generated (either defined by the user via the threshold for amount of nodes or if it is dependent on that the memory of the computer has been used), then the algorithm will increase the MCS threshold according to a pre-defined schema such that the BDD can be generated with the amount of nodes that has been defined. This means that the part calculated using MCUB may increase if too many nodes are generated with the original setting.

The basic formula for calculating the top value for an initiating event with frequency F_{IE} is:

$$F_{Top} = F_{IE} \times (Q_{BDD} + (1 - Q_{BDD}) \times Q_{MCUB}) \qquad (1)$$

Then this formula needs to be extended to cover also sequences (when success events are considered in the MCS list), but this will not be discussed here, as this is not the purpose of this paper.

However, when there are so called mutually exclusive events in the BDD part of the MCS list and in the part that is treated by MCUB the formula will instead be (conservatively):

$$F_{Top} = F_{IE} \times (Q_{BDD} + Q_{MCUB}) \qquad (2)$$

3.2. Problem description

In the previous section we gave a quick overview of the method. The challenge consists in cases like seismic analysis, where the conditional probabilities in the MCS list are very high, to place the cut between the BDD and the MCUB part at a good position. The challenge consists of that when many of the MCSs have got a very high conditional probability, quite many of the MCSs will have a potential impact on the top.

So let us assume that you set the threshold for BDD treatment to 1%, this means that the cutoff may be placed very near the total amount of MCSs (if the MCS list is long and the conditional probability is high). This may in its turn likely trigger the change of the threshold automatically, as too many nodes are generated. This may mean that the fraction of the part of the MCS list that is generated by MCUB is no longer insignificant. As the MCUB is by definition conservative, and its weakness is when there are many high probability events, the precision of the top will be affected.

On the other hand, the precision in the discussed scenarios is not extremely important – because the conditional probability of core melt given the initiating event may be 0.8 or higher – meaning that if the result precision is lowered slightly, it is not necessarily a major concern (but of course we want as high precision as possible). Equation 1 will also ensure that the total result is not greater than the result that would have been calculated using MCUB alone (as MCS BDD generates a higher precision – lower result).

The main issue in the calculation is when there are events that are mutually exclusive in the MCS list. Because then Equation 2 will have to be used. And in this case, it cannot be guaranteed that the approach with BDD and MCUB will actually generate a better approximation than the MCUB on its own.

There are some potential ways to solve this issue:

1. Improve the accuracy by not having any part calculated by MCUB, that is, increase the BDD part.
2. Do not use mutually exclusive events or find a way to not consider them.
3. Find an algorithm that can merge the results using Equation 1 in a better way, which means that management of mutual exclusivity needs to be considered in the different parts of the solution and not only on a top level per sequence and/or initiating event.

The first possible solution is of course a given first attempt; just lower the threshold and increase the amount of BDD nodes that can be generated. If it can include the complete MCS list, or at least the majority of the MCS list, the problem will be solved. The problem is that if the MCSs are long, the generated BDD becomes too big. So, it is not a scalable solution. Even though it can most likely be used on a case-by-case basis and try the optimal setting for the case studied. But we are looking for an approach that is practically possible to use generically, and hence we need to look for something different.

Another option would be to focus on the mutually exclusive events, as in most situations they are not dominant contributors. If there are a few that can be isolated, this would be solvable, as the different partitions could be managed. However, this would introduce a potentially non-conservative treatment (to skip some treatment of mutual exclusive combinations). Therefore, it is not desirable.

The approach studied further is an algorithm based on the type 3 approach.

4. THE BASIS OF THE IMPROVED MCS BDD ALGORITHM

The proposed algorithm addresses the above-described limitation of the current MCS BDD approach, which manifests when the MCUB term contributes non-negligibly to the top value. This limitation is particularly pronounced when the MCUB term contains mutually exclusive (MUX) events, since in such cases the contributions of the BDD-evaluated part and the MCUB term are combined by direct summation rather than via the Min Cut Upper Bound (MCUB) approximation, leading to an overestimation of the top value.

The objective of the proposed method is to combine the BDD-evaluated part and the MCUB term through the MCUB approximation while preserving the logical constraints imposed by the MUX events. Provided that the "MCS threshold" (the value between BDD-evaluated part and MCUB term) for BDD-evaluated part can be maintained at the same level as in the current approach, this formulation will yield a tighter upper bound on the top value.

The basis for the improved algorithm is presented through three examples.

Example 1. An MCS list without frequency events, success modules, or MUX events; the cutoff term contains a single one-event cut set.

Consider an MCS list partitioned into two parts: a part (denoted M, the "BDD-evaluated part") quantified by the BDD algorithm, and a part (denoted P, the "MCUB part") quantified by the MCUB approximation. Assume that the MCUB term P consists of a single one-event cut set P_0 containing the event E_0, so that

$$Q(P) = Q(P_0) = Q(E_0) \qquad (3)$$

This situation is depicted in Figure 2. Aggregating M and P via the MCUB approximation yields

$$\mathrm{TOP} = \mathrm{MCUB}(M, P) = 1 - \big(1 - Q(M)\big)\big(1 - Q(P)\big) = Q(M) + Q(E_0) - Q(M) \cdot Q(E_0) \quad (4)$$

Now suppose that E_0 is independent of every event appearing in the BDD-evaluated part M. Inserting E_0 into the BDD of M produces a new BDD, M', in which E_0 remains independent of all other variables. Taking E_0 as the first pivot variable gives:

$$Q(M') = Q(E_0) + \big(1 - Q(E_0)\big) \cdot Q(M) = Q(M) + Q(E_0) - Q(M) \cdot Q(E_0) \qquad (5)$$

which is identical to $\mathrm{MCUB}(M, P)$. Therefore, when E_0 is independent of the events in M, incorporating E_0 directly into the BDD of M is equivalent to combining M and P through the MCUB approximation.

When E_0 is not independent of the events in M (for instance, when E_0 itself appears in one or more minimal cut sets of M, or is positively correlated with events that do), the occurrence of E_0 raises the conditional probability of M:

$$Q(M \mid E_0) > Q(M) \qquad (6)$$

Consequently,

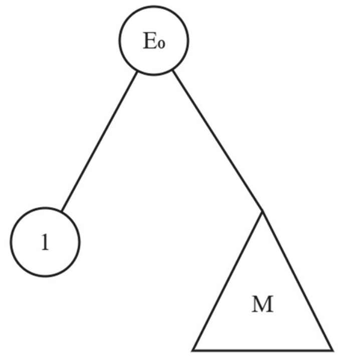
$$Q(M \cap P) = Q(E_0) \cdot Q(M \mid E_0) > Q(E_0) \cdot Q(M) = Q(M) \cdot Q(P), \qquad (7)$$

so the true intersection term is larger than the one implicitly used by MCUB. Substituting this back gives

$$Q(M \cup P) < Q(M) + Q(P) - Q(M) \cdot Q(P) = \mathrm{MCUB}(M, P) \qquad (8)$$

i.e., MCUB overstates the top-event probability. Intuitively, MCUB double-counts the contribution of E_0: once through its role inside M, and again through the cutoff term P, because it fails to recognize that the two parts share (or are correlated through) the same basic event.

Figure 2: Illustration of the approximation in Example 1



Example 2. An MCS list without frequency events, success modules, or MUX events; the cutoff term contains several cut sets.

Consider again an MCS list partitioned into two parts: a part (denoted M, the "BDD-evaluated part") quantified by the BDD algorithm, and a part (denoted P, the "MCUB term") quantified by the MCUB approximation. Suppose that P consists of several cut sets $P_1, P_2, \dots, P_n$. Then

$$Q(P) = \text{MCUB}(P_1, P_2, \dots, P_n) = 1 - \prod_{i=1}^{n}\big(1 - Q(P_i)\big) \quad (9)$$

Aggregating M and P via the MCUB approximation yields

$$\text{TOP} = \text{MCUB}(M, P) = Q(M) + Q(P) - Q(M) \cdot Q(P) \quad (10)$$

For each cut set P_i, define an AND-module, denoted AND-P_i, that groups together all events appearing in P_i. The probability of this module is the product of the mean probabilities of its constituent events:

$$Q(\text{AND-}P_i) = Q(P_i) \quad (11)$$

Next, define an OR-module, denoted OR-P, that groups all of the AND-modules AND-P_1, … ,AND-P_n under a disjunction. The probability of this OR-module is

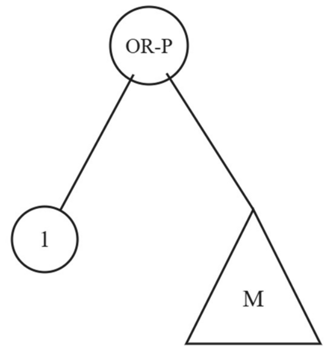
$$Q(\text{OR-}P) = \text{MCUB}(P_1, P_2, \dots, P_n) = Q(P) \quad (12)$$

Figure 3 shows a structure with an OR-P module. Now suppose that OR-P is independent of every event appearing in the BDD-evaluated part M. Inserting OR-P into the BDD of M produces a new BDD, M', in which OR-P remains independent of all other variables. Taking OR-P as the first pivot variable gives

$$Q(M') = Q(\text{OR-}P) + \big(1 - Q(\text{OR-}P)\big) \cdot Q(M) = Q(M) + Q(P) - Q(M) \cdot Q(P) \quad (13)$$

which is identical to $\text{TOP} = \text{MCUB}(M, P)$. Therefore, when OR-$P$ is independent of the events in M, incorporating OR-P directly into the BDD of M is equivalent to combining M and P through the MCUB approximation. In effect, the new BDD M' extends the original BDD M to cover the cutoff term P by regrouping the cut sets $P_1, \dots, P_n$ into the single OR-module OR-P.

Figure 3: Illustration of the approximation using OR-Modules in Example 2



Example 3. An MCS list without frequency events or success modules but containing MUX events.

Consider again an MCS list partitioned into two parts: a part (denoted M, the “BDD-evaluated part”) quantified by the BDD algorithm, and a part (denoted P, the “MCUB part”) quantified by the MCUB approximation. Suppose that the MCS list contains the MUX events A, B, C, a group of events mutually exclusive to each other, we call this group a MUX set $\{A, B, C\}$. Suppose that this is the only MUX set in the MCS list.

Within the BDD-evaluated part M, let C_X denote the subset of the MCS list whose cut sets contain the event X, and let C' denote the subset whose cut sets contain none of A, B, C. The BDD for M can then be constructed by considering the MUX events as follows.

Within the MCUB part P, let P_X denote the subset of the MCS list whose cut sets contain the event X, and let P' denote the subset whose cut sets contain none of A, B, C.

In the current algorithm, the contributions from the different MUX sections, $Q(P_A)$, $Q(P_B)$, and $Q(P_C)$, are first summed directly, and the resulting sum is then combined with $Q(P')$, the contribution of the cut sets containing no MUX event, via the MCUB approximation:

$$Q(P) = \mathrm{MCUB}\big(Q(P'),\ Q(P_A) + Q(P_B) + Q(P_C)\big) \qquad (14)$$

Because the MCUB part contains MUX events, $Q(P)$ must be combined with $Q(M)$ by direct summation rather than by MCUB, giving

$$\mathrm{TOP} = Q(M) + Q(P) \qquad (15)$$

When the MCUB part $Q(P)$ contributes non-negligibly to the top value, this direct summation leads to an overestimate. Intuitively, the direct sum double-counts at minimum the contribution $Q(C_X) \cdot Q(P_X)$ for each MUX event $X \in \{A, B, C\}$.

To address this problem, we introduce four OR-modules, OR-P_A, OR-P_B, OR-P_C, and OR-P', and extend the BDD M by incorporating these modules at the appropriate positions. Concretely, D_X denotes the union of C_X and OR-P_X for each $X \in \{A, B, C\}$, and D' denotes the union of C' and OR-P'. The extended BDD M' can be constructed by defining nodes for *A, B, C*, and linking them to corresponding children.

When several MUX sets are present, each MUX section produces a single OR-module, which is then incorporated into the corresponding cut set list of that MUX section within the BDD part.

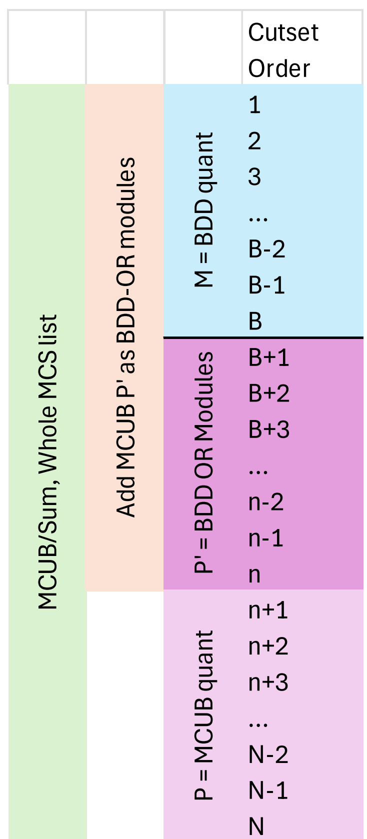
5. THE IMPROVED MCS BDD ALGORITHM

The improved MCS BDD algorithm implemented in RiskSpectrum builds on the same theoretical basis as has been explained in previous papers [1, 2, 3, 4]. But we extend the concept of splitting the MCS list into one part treated by MCS BDD, and one part treated by MCUB. In our improved approach, we introduce a third part in which the content of the MCSs are replaced by OR modules (calculated using MCUB). This means that the modules can be incorporated into the BDD structure.

In case there are events that are considered as MUX versus other events, they are not included into the OR modules, but such events are considered separately. Hence, for each combination of MUX events in the MCS list, an OR module is created – which enables the inclusion of such modules in appropriate places in the BDD and thereby manage the problem with mutual exclusive events in the "BDD part" and in the "MCUB part".

We still keep one part that is solely managed by MCUB, where the previously described issue still can remain – but we can enforce that this part will never be significant in the calculation by making it insignificant to the top. This part is kept just to ensure scalability and speed.

Figure 4: Representation of how the MCS list is split into one part calculated using BDD (above MCS no. B), one part where MCUB is put into OR-Modules included in the BDD (between B and n) and MCUB part below n



The process described above also needs to consider other elements that require a split of the MCS into different parts – like frequency events and success modules. This is however not described in this paper.

6. CONCLUSION

This paper outlines a corner case situation which can challenge the accuracy of the MCS BDD approach implemented in RiskSpectrum, when the algorithm cannot generate sufficient number of nodes in the BDD due to time or memory constraints.

The issue can in such situations mean that the precision using the MCS BDD is decreased compared to the normal MCUB approach in RiskSpectrum. There are two conditions necessary to trigger this effect. First, there are very many MCSs with high conditional probability such that the top frequency value is very close to the initiating event frequency. Second, there are mutually exclusive events defined within the analysis. Such situations can occur in seismic analysis, typically in the last interval when the core damage frequency is close to the hazard frequency.

The issue is caused by the fact that the MCS list is split into one part that is using BDD and one part that is using MCUB. The issue lies in how these two groups of the MCS list can be combined, and specifically when the MCS list contains events that are defined as mutually exclusive.

In this paper we present a solution to the corner case, in which we improve the treatment of how the BDD and MCUB part are combined.

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