Efficient Reliability Demonstration using the Probability of Test Success and Bayes Theorem

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Abstract: In order to demonstrate the reliability of a component, reliability engineers are often faced with multiple challenges. On the one hand the budget for testing is limited and on the other hand, the demonstration needs to be done as quick and with the most precise statistical information possible. To address these challenges, the concept of Probability of Test Success was developed [1]–[3]. It enables the objective assessment of tests with regard to their chance of a successful reliability demonstration and thus the ability to objectively compare the test configurations as well as the planning of expenditure and cost estimation. Secondly, approaches have been developed which, by means of Bayes' theorem, use available prior knowledge to correct the information obtained from the tests and thus reduce expenditures [4]–[6]. However, the combination of the Probability of Test Success and Bayes' Theorem to plan efficient reliability demonstration tests has not been addressed so far. Since both approaches make use of the available prior knowledge, it is analyzed how reliability demonstration tests can be planned using the Probability of Test Success in combination with the reliability methods of Bayes' Theorem. The combined use of the two approaches reveals a considerable advantage. Not only is it possible to select the optimal test according to the boundary conditions by means of the prediction of demonstration success. In addition, the integration of prior knowledge by means of Bayes' theorem enables an additional significant reduction of expenditure, which may result in an even higher chance of success. The presented approach is capable of planning and assessing failure-based tests as well as any censored and failure-free tests. Procedures for the most typical scenarios are introduced and the advantages as well as disadvantages for the most common cases are worked out. The approach is additionally illustrated by way of exemplary cases. The results show that the demonstration of reliability and its necessary planning of tests benefit greatly from the proposed combination of the two approaches.

1. INTRODUCTION

One of the main requirements of a product is to provide its functions over the desired lifetime in accordance with the respective boundary conditions. For new technologies and new products with a high multitude of functions, the importance of reliability grows significantly. In order to ensure and demonstrate the integrity of the product in terms of reliability, the product needs to be physically tested. The planning of such reliability demonstration tests is challenging, since a great variety of test types and configurations are available and the test has to comply to expenditure constraints. However, an intelligent planning takes advantage of the constellation of the individual products characteristics and the reliability to be demonstrated by making use of available prior knowledge. In order to make the identification and planning of such efficient test plans possible, the Probability of Test Success (P_{ts}) was established by Dazer et al. [1], [2]. Grundler et al. [3], [7]–[9] defined it as the statistical power of the reliability demonstration test and its configuration. It is therefore possible to objectively assess and compare the available reliability demonstration tests with regard to their chance of a successful reliability demonstration in addition to their feasibility regarding the constraints. Applicability has already been demonstrated in [2], [8]–[11], as has the transferability to accelerated tests with different lifetime models in [12]–[16]. Even tests for complex systems with several system levels can be planned efficiently on an objective level [7], [8], [17].

The reliability information obtained from the realized test, can be enriched by combining them with prior knowledge using Bayes' theorem [6], [18], [19]. To already make use of such prior knowledge in test planning phases, several approaches can be found in the literature. For example, Kleyner et al. [20], [21], and Krolo et al. [22]–[25] both use prior knowledge for planning tests and demonstrating reliability using zero failure tests, also known as Success-Run tests (SR tests). Those approaches are mainly based on the planning approach presented by Beyer and Lauster [24], [26], [27], which uses a single value of reliability as prior knowledge and transforms it into an equivalent SR test. Using beta distributions as prior knowledge about reliability can also be used for system reliability demonstration [28], [29] as well as for new technologies making use of high precision lifetime calculations [30], [31]. However, the planning of failure-based tests is often neglected if the use of Bayes' theorem is concerned. In addition, the combination of Bayes' theorem and the concept of Probability of Test Success has not been studied yet, but promises significant improvements in the identification of efficient reliability demonstration tests. This paper therefore presents an approach which makes use of the combined use of Bayes' Theorem and Probability of Test Success which enables the identification of the most efficient test (assessment through P_{ts}) while ensuring a minimum in expenditure (by additionally using Bayes' Theorem). Typical scenarios of prior knowledge and tests are analysed.

2. PRIOR KNOWLEDGE

Products are never entirely new products, there are at least some similar components which are already in use in another product. So, product development takes place with a certain degree of adoption of a previous product generation, of similar products or similar concepts [32]–[35]. Therefore, some kind of prior knowledge about the product behaviour is available. For reliability demonstration, some prior knowledge about the failure behaviour may be available, for example by the conducted lifetime calculations during development of the geometry and gestalt. In addition, some prototype tests may be performed or field data from a preceding product is known. In conclusion, sources of prior knowledge could be:

- Historical data
- Field monitoring of similar products
- Prototype tests
- Lifetime calculations
- Expert knowledge
- Reliability tests of similar products
- Reliability tests of predecessor products
- Lifetime tests of the components
- Tests of the supplier

This information usually is available in different forms. The tests of prototypes, the reliability tests, the tests of the supplier and the components testing may yield failure times and therefore an estimation of the failure distribution. If only censored data is available, only an estimation of the reliability itself can be made. Expert knowledge may not be translated into a quantifiable measure without effort [36], [37]. However, the main sources, data and information of prior knowledge, can be transformed into a reliability information of one of two types: On the one hand, if no failures can be attested or assigned to the prior knowledge, an information about the reliability itself should be available, e.g., a beta distribution [38]. On the other hand, if failures were observed, an information about the failure distribution can be obtained, e.g., a Weibull distribution [39]. Hence, the two types of prior knowledge about the reliability are the following:

1. Prior knowledge stemming from a Success-Run test (SR test): a distribution of reliability as the confidence distribution, e.g., a beta distribution for a specific lifetime of the product
2. Prior knowledge stemming from an End-of-Life test (EoL test): a sample of failure times (with censoring), or a failure distribution, e.g. a Weibull distribution and a sample size, which can be made responsible for estimating the failure distribution

These two types of prior knowledge need to be used in different ways in reliability demonstration test planning, according to the test type which shall be planned.

3. USING PRIOR KNOWLEDGE WITH THE PROBABILITY OF TEST SUCCESS

In order to assess reliability demonstration tests using the P_{ts}, prior knowledge about the reliability at test suspension must be available for SR tests. For the planning of EoL tests, prior knowledge about the failure distribution must be available. Of course, knowledge about the failure distribution can also be used for the planning of SR tests and is even needed, if the suspension time of the specimen shall be another than the to be demonstrated lifetime. Prior knowledge about the failure distribution therefore represents the optimal starting point for reliability demonstration test planning using the P_{ts} [40]. Since the uncertainty of the prior knowledge in terms of its initial sample size shows a significant effect on the P_{ts} [41], it shall be used herein. The general procedure for planning reliability demonstration tests using the P_{ts} is the following:

1. Gather prior knowledge about failure distribution (assessing EoL and SR tests) or reliability distribution at test suspension time (sole assessing of SR tests). For sources, see section 2.
2. Identify constraints in expenditure: e.g., financial budget will constrain both available sample size and test time. Restrictions in time constrain the available time for testing.
3. Calculate P_{ts} for all test scenarios, which may comply with the constraints:
    a. EoL tests in uncensored and censored fashion for several feasible sample sizes
    b. SR test for sample size according to the reliability requirement, as well as several lifetime ratios as long as they are feasible
4. Choose test which yields the highest number in P_{ts}. Alternatively: Calculate P_{ts} cost ratio or P_{ts} time ratio and choose test with the highest number which still follows a minimum in P_{ts} (e.g., 70 %).

If accelerated tests are desired, the lifetime model needs to be available for the SR test as well as the EoL test. However, the EoL test can also be used to estimate the lifetime model at the same time the lifetime quantile is estimated for reliability demonstration. The use of the P_{ts} in this specific scenario can be found in [14], [42]. Step 3 and 4 can be combined if an applicable optimization algorithm is used for the identification of the optimal test in the individual scenario. However, the algorithm needs to be trusted for it to actually find the test plan with the desired conditions.

3.1. The Probability of Test Success

The Probability of Test Success P_{ts} is defined as the statistical power of a reliability demonstration test. Since the hypotheses of such a test are always the same, the new term was first coined in [17] for convenience and better understanding in analogy to the confidence level C, which is already very well understood and used in the context of reliability engineering. The hypotheses are defined via the lifetime quantiles t_{R_r} at required reliability R_r and the required lifetime quantile t_r for EoL tests [3], [7], [8], [41]:

$$H_0: t_{R_{\mathrm{r}}} < t_{\mathrm{r}} \tag{1}$$

$$H_1: t_{R_{\mathrm{r}}} \geq t_{\mathrm{r}} \tag{2}$$

Since SR tests only yield an estimate of the reliability itself via e.g., a binomial or beta distribution, the hypotheses for SR test are defined as follows:

$$H_0: R(t_{\mathrm{r}}) < R_{\mathrm{r}}(t_{\mathrm{r}}) \tag{3}$$

$$H_1: R(t_{\mathrm{r}}) \geq R_{\mathrm{r}}(t_{\mathrm{r}}) \tag{4}$$

Using the required confidence level $C_r = 1 - \alpha$, the P_{ts} is the complement of the statistical error β (see e.g., [43] and [3]) of the test

$$P_{\mathrm{ts}} = 1 - \beta \tag{5}$$

and can be calculated for EoL tests, using the null and alternative distribution f_{H_0} and f_{H_1} of t_{R_r}, which represent the distribution of the lifetime quantile under validity of the respective hypothesis [3], [8]:

$$C_{\mathrm{r}} = \int_{-\infty}^{t_{\mathrm{crit}}} f_{H_0}\, \mathrm{d}t_{R_{\mathrm{r}}} \tag{6}$$

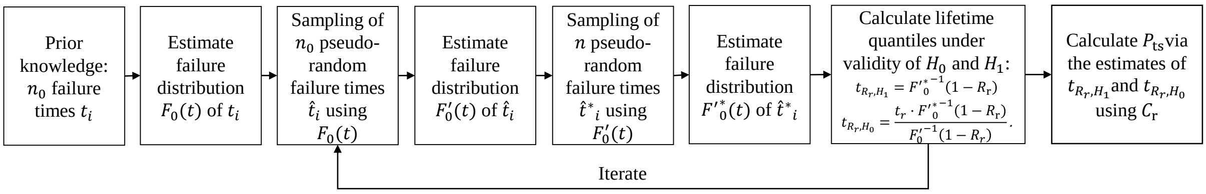
$$P_{\mathrm{ts}} = \int_{t_{\mathrm{crit}}}^{+\infty} f_{H_1}\, \mathrm{d}t_{R_{\mathrm{r}}} \tag{7}$$

The closed analytic form for calculating the P_{ts} of an SR test using the binomial approach (see [3], [8], [17]), cannot be used here unaltered, since the uncertainty of prior knowledge needs to be considered [41]. In order to account for the uncertainty a bootstrap approach is presented in the following.

3.2. Calculating the Probability of Test Success

The general procedure for calculating the P_{ts} while considering the uncertainty of prior knowledge about the failure distribution is described in [41]. An abbreviated version of the therein used double bootstrap-algorithm for EoL tests is depicted in Fig. 1.

Figure 1: Double Bootstrap Algorithm for the Calculation of P_{ts} of an EoL Tests [41]



In the first step, pseudo-random failure times are drawn from the known failure distribution according to the initial sample size $\boldsymbol{n_0}$ of the prior knowledge. These failure times are then used to estimate the parameters of the failure distribution in order to estimate the uncertainty of the parameters of the prior knowledge itself. Using these parameters, additional pseudo-random failure times are drawn, this time, according to the sample size n of the tests to be analysed. These failure times are then used to estimate the failure distribution again. According to this distribution, the appropriate lifetime quantiles under validity of H_0 and H_1 are calculated. After iterating these steps multiple times (e.g., 10,000 times) the generated lifetime quantiles can be used to calculate the P_{ts} using the null and alternative distribution f_{H_0} and f_{H_1} and the equations 6 and 7.

If no failure distribution is known from prior knowledge but instead prior knowledge from an SR test about the reliability itself, the binomial distribution (sample size n and number of failures k) can be translated into a beta distribution with parameters A and B that describe the distribution of reliability, via $A = n - k$ and $B = k + 1$ for the corresponding relevant lifetime. If the runtimes of the specimen differ from the relevant lifetime (e.g., the required lifetime), a Weibull shape parameter is needed for example, see e.g., [44]. In rare cases, due to a lack in documentation, it could be the case that only a single value of the reliability for the corresponding lifetime is known. In this case, the approach of [27] can be used to still derive a beta distribution.

Since the reliability is described by the beta distribution and the SR test result is described by a binomial distribution, the P_{ts} can be calculated for an SR test using a beta-binomial distribution

$$\begin{aligned} P_{\text{ts}} &= \sum_{i=0}^{k} \binom{n}{i} \frac{\beta(i+B_0,\ n-i+A_0)}{\beta(B_0, A_0)} \\ &= 1 - \frac{n \cdot \beta(A_0+n-k-1, B_0+k+1)\Gamma(n)\,{}_3F_2(1, B_0+k+1, -n+k+1; k+2, -A_0-n+k+2; 1)}{\beta(B_0, A_0)\beta(n-k, k+2)\Gamma(n+2)} \end{aligned} \tag{8}$$

and enables an analytic expression and calculation if the prior knowledge about the reliability is available via a beta distribution from another SR test with parameters A_0 and B_0 in order to account for the uncertainty of the reliability at suspension time of the SR test which has a sample size of n and a maximum number of allowed failures of k. Here $\beta(A, B)$ is the complete beta function (see e.g., [38]) with parameters A and B, $\Gamma(\cdot)$ is the gamma function (see e.g., [45]) and ${}_3F_2(p_1, p_2, p_3; p_4, p_5; p_6)$ is the generalized hypergeometric function (see e.g., [46]) which takes six values as input: p_1 to p_6. This equation 8 allows for a calculation of the P_{ts} while considering the uncertainty of prior knowledge if only a beta distribution is known from prior knowledge. This is an advantage for the practical application, since no prior knowledge about the failure distribution is to be known, which previous approaches did require. In addition Equation 8 should coincide with the bootstrap approach presented in [41], if the beta distribution used in Equation 8 is generated from the failure distribution of prior knowledge using a bootstrap approach, e.g., the one presented in [30].

The use of the different types of prior knowledge for the calculation of the P_{ts} is summarized in Table 1. It has to be noted, that due to the type of available prior knowledge, an EoL test cannot be planned using the P_{ts} if only an SR test is available as prior knowledge. There needs to be an estimate about the failure distribution.

Table 1: Use of the Two Types of Prior Knowledge for P_{ts} Calculation for the Two Types of Tests Which can be Assessed or Planned

		Test type to be panned / assessed	
		SR test	**EoL test**
Type of prior knowledge	**SR test** (Beta distribution)	Beta-binomial distribution: $P_{\text{ts}} = \sum_{i=0}^{k} \binom{n}{i} \frac{\beta(i + B_0,\ n - i + A_0)}{\beta(B_0,\ A_0)}$	Not applicable: Does require knowledge about failure distribution
	EoL test (failure times or failure distribution)	Generate beta distribution about reliability at test suspension time (see [30], [41]) and use beta-binomial distribution: $P_{\text{ts}} = \sum_{i=0}^{k} \binom{n}{i} \frac{\beta(i + B_0,\ n - i + A_0)}{\beta(B_0,\ A_0)}$ Alternatively: simulate test via bootstrap approach of [41]	Use double bootstrap algorithm of Fig. 1 (see also [41])

Reliability engineers may find themselves in a situation, where although prior knowledge is available, the underlying sample size of this information may be unknown. This could be, for example, if the documentation is insufficient or the supplier only provides such information. For such uncertain sample size information, one can assume a conservative estimate: e.g., $n = 3$ for failure distributions and e.g., $n = ln(0.5)/ln(R_{\text{p}})$ (cf. [27]), for information about the reliability R_{p} and subsequent generation of a beta distribution using $A = n$, $B = 1$. Although being very conservative, it most certainly is more representative of the actual information available, than neglecting the uncertainty, which translates to an assumed sample size of infinity. The consideration of the uncertainty of prior knowledge in P_{ts} calculation generally leads to a decreased value in P_{ts}.

Ultimately, the P_{ts} allows for a determination of the required sample size of the EoL test if a certain minimum value of P_{ts} shall be achieved by it. For an SR test, the P_{ts} enables the assessment of the much less configurable configuration of it. Due to the sample size being coupled to the reliability requirement, only permissible failures, used lifetime ratios and acceleration in terms of higher loads allow a modified sample size. The P_{ts} sheds light on these changes in terms of the expected outcome of the test. In Addition, the comparison of all the feasible tests with regard to cost and time constraints, allows for often more efficient EoL tests, when someone traditionally would not have considered such failure-based tests, due to the lack of an appropriate planning instrument like the P_{ts}.

4. USING PRIOR KNOWLEDGE WITH BAYES THEOREM FOR RELIABILITY DEMONSTRATION

Since two types of prior knowledge can be available to be taken into account via Bayes' Theorem to enrich the obtained information of a reliability demonstration test, a distinction between these types is required. Bayes' Theorem describes conditional probabilities and is mainly used to combine information or distributions to obtain more precise statements about the matter at hand [18]. The prior distribution (e.g., about reliability or lifetime quantile) gets updated with information from a current test to form the posterior distribution which holds both information [47], [48].

4.1. SR Test as Prior Knowledge

The reliability information of an SR test can be translated into a beta distribution. Since the binomial distribution of the to be planned SR test and the beta distribution of the prior knowledge are conjugate distribution on terms of Bayes' Theorem [49], the posterior distribution of the combined information of the planned as well as the SR test of prior knowledge is the following [27], [30], [31]:

$$f_{\text{post}}\,(R) = \mathcal{B}(n - k + A_0, k + B_0) \qquad (9)$$

With $\mathcal{B}(A, B)$ being the probability density function of the beta distribution or incomplete beta function with parameters A and B.

In order to use the prior knowledge of an SR test in the form of a beta distribution to enrich an EoL test, a beta distribution of reliability at the required lifetime of the EoL test has to be obtained. This can be done by performing a bootstrap on the EoL test data and fitting a beta distribution [17], [30], [31], see Fig. 1. Opposed to estimating lifetime quantiles, reliabilities should be estimated in order to fit the beta distribution. The beta distribution of the EoL test with parameters A and B can then be combined with the beta distribution of prior knowledge using Bayes' Theorem [31]:

$$f_{post}\,(R) = \mathcal{B}(A + A_0 - 1, B + B_0 - 1) \quad (10)$$

This confidence distributions of reliability of equations 9 and 10 can be used for the planning of the SR test by finding the sample size n and maximum number of allowed failures k so that the reliability requirement is met by:

$$\int_{R_\mathrm{r}}^{1} f_\mathrm{post}(R)\mathrm{d}R \overset{?}{\geq} C_\mathrm{r} \quad (11)$$

4.2. EoL Test as Prior Knowledge

To make use of prior knowledge of an EoL test in the form of a failure distribution with sample size or failure times of the EoL test, the same bootstrap procedure can be used in order to obtain a beta distribution with parameters A_0, B_0 for combination with the result of an SR test (see Fig. 1).
If an EoL test is to be combined with an EoL test, it has to be distinguished between the case of prior knowledge being the whole sample with failure times or only the information about the estimated failure distribution with a sample size.

If the sample of the EoL test of prior knowledge is known (sample size n_0), its likelihood function L_0 can be used as the prior distribution in the maximum a posteriori probability estimation (MAP) [50] with the likelihood of the sample of the new EoL test L_1 as follows:

$$L_\mathrm{post} \sim L_1 \cdot L_0 = \prod_{i=1}^{n} f(t_i) \cdot \prod_{j=1}^{n_0} f(t_j) = \frac{b}{T} \prod_{i=1}^{n} \left(\frac{t_i}{T}\right)^{b-1} e^{-\left(\frac{t_i}{T}\right)^b} \cdot \prod_{j=1}^{n_0} \left(\frac{t_j}{T}\right)^{b-1} e^{-\left(\frac{t_j}{T}\right)^b} \quad (12)$$

The likelihoods are here exemplarily displayed for a two parameter Weibull distribution with scale parameter T and shape parameter b for uncensored samples. Naturally the likelihoods can also be used for arbitrarily censored samples. The MAP represents the application of Bayes' Theorem by incorporating the likelihood of the sample from prior knowledge as the prior distribution of the distribution parameters which get updated by the likelihood of the sample of the current EoL test. To find the new, updated parameter estimates of the failure distribution, the posterior distribution does not need to be normalized. This is why Equation 12 is written as a proportional statement rather than an equation. Since the application of Bayes' Theorem here results in a multiplication of the two likelihoods, the two samples of failure (and censored) times can simply be concatenated and used for a Maximum Likelihood Estimation (MLE) of the failure distribution. In other words, Bayes' Theorem here acts as if both samples were drawn in the same experiment.

If the prior knowledge of the EoL test is available as a failure distribution with known sample size, this simple combination cannot be used without an additional step. The proposed step here is very pragmatic, but should approximate well and is similar to the approach used in [3] for the generation of a synthetic sample for analytical calculation of the P_ts. The values t_j^* of the here generated synthetic sample are calculated according to the distributions of order statistics [51]. Using the distribution of failure probability of the order statistics, just like in the approach for calculating beta-binomial confidence bounds, see [52]. For a known sample size n_0 of prior knowledge about the failure distribution $F_0(t)$, the n_0 synthetic failure times t_j are calculated as

$$t_j^* = F_0^{-1}\left(\mathcal{B}^{-1}(0.5; j, n_0 - j + 1)\right) \approx F_0^{-1}\left(\frac{j-0.3}{n_0+0.4}\right). \quad (13)$$

Here, the well-known approximation to the median of the beta distribution of Benard [53] is used.
$\mathcal{B}^{-1}(q; A, B)$ being the inverse of the beta distribution, its quantile function for quantile q with parameters A and B. If a two parameter Weibull distribution is given by prior knowledge with scale parameter T_0 and shape parameter b_0, above equation becomes

$$t_j^* \approx T_0 \left(-\ln\left(1 - \frac{j-0.3}{n_0+0.4}\right)\right)^{1/b_0} \quad (14)$$

with $\ln(\cdot)$ being the natural logarithm.

If censoring was present in the sample used for estimation of the failure distribution of prior knowledge and the censoring scheme is known, it best be accounted for accordingly in the synthetic sample.
The use of prior knowledge for the two types and the two test types is summarized in Table 2.
If multiple sources of prior knowledge are available, they need to be combined using Bayes' Theorem. Multiple SR tests need to be combined into one beta distribution using Equation 9 or 10. Multiple EoL tests need to be combined using the MAP of Equation 12 with Equation 13 and 14 if needed.
If both an EoL test as well as an SR test are available as prior knowledge, they should be used both according to Table 2, see also section 5.

Table 2: Use of the Two Types of Prior Knowledge for Consideration with Bayes Theorem

		Test type the prior knowledge shall be combined with	
		SR test	**EoL test**
Type of prior knowledge	**SR test** (Beta distribution)	Combine beta distribution of prior knowledge (parameters A_0 and B_0) with sample size n and allowed failures k of SR test: $A_{\text{post}} = n - k + A_0$ $B_{\text{post}} = k + B_0$	Generate beta distribution about reliability at test suspension time (see [30], [41]) from EoL test sample to get A and B and apply Bayes Theorem to get combined information in one beta distribution: $A_{\text{post}} = A + A_0 - 1$ $B_{\text{post}} = B + B_0 - 1$
	EoL test (failure times or failure distribution)	Generate beta distribution about reliability at test suspension time (see [30], [41]) from prior knowledge EoL test to get A_0 and B_0 and apply Bayes Theorem: $A_{\text{post}} = n - k + A_0$ $B_{\text{post}} = k + B_0$	Concatenate both EoL samples (sample from prior knowledge or synthetic sample from prior knowledge and sample from current EoL test) to make failure distribution estimation. Generate beta distribution about reliability using bootstrap approach of [30] if desired.

5. USING BOTH PROBABILITY OF TEST SUCCESS AND BAYES THEOREM FOR RELIABILITY DEMONSTRATION AND TEST PLANNING

The approach of test planning using the P_{ts} as well as the approach of additionally using the available prior knowledge via Bayes' Theorem are very valuable for reliability demonstration. In order to benefit from both approached to ensure a most efficient reliability demonstration, the two approaches need to be combined. The approach for this combined use of the concept of P_{ts} and Bayes' Theorem is presented here and an analysis of the cases for planning EoL tests or SR tests and doing so using prior knowledge about the failure distribution (e.g., from an EoL test) or prior knowledge about the reliability itself (e.g., from an SR test) is conducted.

In the following, no applicability analysis of the prior knowledge is considered, which means, that the prior knowledge and information is assumed to be applicable in all cases and can be used to its full extend. In other words: there is no withdrawal of the prior knowledge taking place if the results of the conducted test correspond to the null hypothesis (reliability requirement is not met despite prior knowledge telling the opposite). It is always being used by means of Bayes' Theorem, no matter the validity of the hypotheses. Some approaches for an applicability analysis do exist in the context of Bayes methods in reliability engineering, see e.g., [25], [36], [37], [54], [55]. However, these approaches do not facilitate a comprehensive procedure for an objective assessment in this regard and show some conceptual disadvantages in the formulation of the applicability factor. Since the focus of the work presented here is solely on the combination of P_{ts} and Bayes' Theorem, these approaches shall not be used here. However, they might be integrated in future studies.

5.1. Approach

The approach combines the presented procedures of section 3 and 4. For planning SR tests, it is a two-step approach, first, the prior knowledge is combined with the hypothetically to be obtained reliability information with regards to the sample size. After the required sample size is determined, the P_{ts} is calculated based on the prior knowledge and the calculated required sample size. For EoL tests the procedure is similar, however, the integration of prior knowledge by means of Bayes' Theorem is done during P_{ts} calculation. In the following, the focus is on the calculation of the P_{ts} itself. The procedure for identification of the optimal test is illustrated in section 6.

5.1.1. Case 1: Planning an SR test with an SR test as prior knowledge

For the calculation of the P_{ts} of an SR test while using prior knowledge stemming from an SR test, the procedure is the following:

1. Combine beta distribution of prior knowledge with binomial distribution of the SR test, see Equation 9.
2. Calculate required sample size of the SR test, which is required to meet the reliability requirement, if the test is successful, using Equation 11:
   Find n_{r} for which $C_{\text{r}} - \int_{R_{\text{r}}}^{1} f_{\text{post}}(R, n_{\text{r}})\,\mathrm{d}R$ is minimal
3. Calculate P_{ts} using the cumulative beta-binomial distribution of Equation 8.

If test times other than the required lifetime are possible, an analysis of lifetime ratios with regard to the P_{ts} can de done. The optimal test in this case is the one with the maximum in P_{ts}, while still feasible in terms of cost and time constraints. However, in order to use lifetime ratios a Weibull shape parameter is required.

5.1.2. Case 2: Planning an EoL test with an SR test as prior knowledge

To calculate the P_{ts} of an EoL test, prior knowledge about the failure distribution is necessary. Prior knowledge about the reliability itself from an SR test (e.g., in the form of a beta distribution) is insufficient. However, if the failure distribution can be estimated, but shall not be used via Bayes' Theorem (e.g., expert knowledge), the EoL test can still be planned using the P_{ts} while incorporating the prior knowledge of the SR test, see case 6.

5.1.3 Case 3: Planning an SR test with an EoL test as prior knowledge

The procedure is very similar to the one of case 1. However, the beta distribution of prior knowledge for the appropriate lifetime needs to be calculated from the prior knowledge of the EoL test, first.

1. Calculate beta distribution of prior knowledge using bootstrap approach of section 3 (see Fig. 1 and [30]). This bootstrap procedure needs to be of parametric form, if the failure distribution is given and can be of non-parametric form, if the sample from the EoL test is given as prior knowledge.
2. Combine beta distribution of prior knowledge with binomial distribution of the SR test, see Equation 9.
3. Calculate required sample size of the SR test, which is required to meet the reliability requirement, if the test is successful, using Equation 11:
   Find n_{r} for which $C_{\text{r}} - \int_{R_{\text{r}}}^{1} f_{\text{post}}(R, n_{\text{r}})\,\mathrm{d}R$ is minimal
4. Calculate P_{ts} using the beta-binomial distribution of Equation 8.

Sample sizes that differ to the calculated required one of step 3 may be possible, if lifetime ratios are analysed and feasible. The optimal SR test is the one with maximum P_{ts} within time and cost constraints. Alternatively, the one fulfilling a minimum value of P_{ts} (e.g., $P_{\text{ts}} > 80\,\%$) while minimizing time and/or cost.

5.1.4. Case 4: Planning an EoL test with an EoL test as prior knowledge

For calculating the P_{ts} of the EoL test, samples are drawn from the known failure distribution and lifetime quantiles are calculated under validity of both hypotheses of Equation 1 and Equation 2, cf. [41]. However, in order to incorporate the prior knowledge via Bayes' Theorem, in each step of failure distribution estimation (of the samples of size n of the to be planned EoL test), the sample of prior knowledge needs to be additionally considered. The procedure is the following:

1. If not already available from prior knowledge: Estimate failure distribution F_0 parameters, e.g., T_0 and b_0 of a Weibull distribution, of the sample of size n_0 of prior knowledge
2. Iterate:
    a. Draw sample of size n_0 from failure distribution F_0 of prior knowledge
    b. Estimate failure distribution F_0' using sample of step 2 a.
    c. Draw sample of size n of the to be planned EoL test from failure distribution $F_0{}'$ and use censoring scheme of the test, if necessary.
    d. Concatenate the drawn sample of step 2 c by the initial sample of prior knowledge (of step 1). If no failure times of prior knowledge are available, create synthetic ones using Equation 13 or Equation 14 and F_0.
    e. Estimate failure distribution F' using combined sample of step 2 d and calculate lifetime quantile $t'_{R_r} = F'^{-1}(1 - R_\mathrm{r})$.
    f. Calculate lifetime quantile under validity of H_0 and H_1, see [41]: calculated value of t'_{R_r} is valid for H_1. For validity under H_0 the value needs to be shifted by multiplying it by $\frac{t_r}{F_0'^{-1}(1-R_\mathrm{r})}$.
3. Calculate P_ts using null and alternative distribution with Equation 6 and Equation 7.

Using the values of P_ts with regard to the sample size n for the uncensored EoL test as well as for different censoring schemes can be very helpful to identify the most efficient EoL test for reliability demonstration. The test cost and test time can be calculated using the failure times of step 2 c and the sample size in order to identify feasible test configurations.

5.1.5. Case 5: Planning an SR test with an EoL test and an SR test as prior knowledge

If both a prior knowledge from an EoL test as well as from an SR test are available, the appropriate cases above need to be combined. For the planning of an SR test, the beta distribution of reliability needs to be calculated from the EoL test for the appropriate lifetime by means of the proposed bootstrap approach (see case 3). It then needs to be combined with the beta distribution of the SR test of prior knowledge using Bayes' Theorem of Equation 10. The procedure of case 1 can subsequently follow using the combined prior knowledge as a single beta distribution.

5.1.6. Case 6: Planning an EoL test with an EoL test and an SR test as prior kowledge

If an EoL test is to be planned, the (synthetic) sample of the EoL test of prior knowledge needs to be considered using the MAP of case 4 and in addition, the prior knowledge of the SR test needs to be considered. In order to consider the distribution of reliability of the SR test, the more general formulation presented in [31] can be used:

$$f(R) = R^{\sum_i \left(\frac{t_i}{t_\mathrm{r}}\right)^b} \cdot \prod_j \left(1 - R^{\sum_j \left(\frac{t_j}{t_\mathrm{r}}\right)^b}\right) \Big/ \int_0^1 R^{\sum_i \left(\frac{t_i}{t_\mathrm{r}}\right)^b} \cdot \prod_j \left(1 - R^{\sum_j \left(\frac{t_j}{t_\mathrm{r}}\right)^b}\right) \mathrm{d}R \tag{15}$$

Here t_i are the lifetimes of the suspended specimen of the SR test, t_j are the lifetimes of the failed specimen and b is the Weibull shape parameter of the prior knowledge from the EoL test. If this equation now gets evaluated at R_r and t_r is considered as the argument of the function (instead of R), the distribution $f(t_{R_\mathrm{r}})$ of the lifetime quantile t_{R_r} stemming from the prior knowledge of the SR test is established:

$$f(t_{R_r}) = R_\mathrm{r}^{\sum_i \left(\frac{t_i}{t_{R_\mathrm{r}}}\right)^b} \cdot \prod_j \left(1 - R_\mathrm{r}^{\sum_j \left(\frac{t_j}{t_{R_\mathrm{r}}}\right)^b}\right) \Big/ \int_0^{+\infty} R_\mathrm{r}^{\sum_i \left(\frac{t_i}{t_{R_\mathrm{r}}}\right)^b} \cdot \prod_j \left(1 - R_\mathrm{r}^{\sum_j \left(\frac{t_j}{t_{R_\mathrm{r}}}\right)^b}\right) \mathrm{d}t_{R_\mathrm{r}} \tag{16}$$

This distribution of the lifetime quantile t_{R_r} can then be combined with the null and alternative distribution f_{H_0} and f_{H_1} of the bootstrap approach of the calculation of the P_ts, see step 2 f of case 4 and Fig. 1 as well as section 3 (cf. [8], [41]). Since the calculations of Equations 6 and 7 take place using the law of large numbers in empiric non-parametric form, the values of t_{R_r} under validity of the null hypothesis and the alternative hypothesis represented by f_{H_0} and f_{H_1} need to be weighted by $f(t_{R_\mathrm{r}})$ (Equation 16). For the weighting to take place, the denominator of Equation 16 is not needed and the numerator is sufficient for practical calculation.

It has to be noted, that due to the worst-case approach of the SR test, the combination of the distribution of the lifetime quantile and the distributions f_{H0} and f_{H_1} of the EoL (either already combined with prior knowledge from another EoL test or not) could lead to a more pessimistic distribution in terms of lifetime. This is because $f(t_{R_\mathrm{r}})$ of the SR test is highly right-skewed. It is effectively the application of Bayes' Theorem.

5.2. Comment on Conducting and Evaluating Planned Tests

The aforementioned procedures describe how the P_{ts} can be calculated while considering uncertain prior knowledge and additionally use this prior knowledge via Bayes' Theorem. If a test is planned using these procedures, the actual test that is being conducted, needs to make use of the Bayes' Theorem. For the SR tests, this is already done by reducing the required sample size by prior knowledge, so no additional steps are needed after conducting the test. For an EoL test however, it is very important, that, after conducting the test, the failure distribution estimation and confidence interval estimation is done with the combined sample of the test and the prior knowledge. If an SR test is available as prior knowledge, the EoL test should additionally be enriched by this information. In order to do so, the (combined) sample of the EoL test needs to be translated into a beta distribution using the bootstrap approach of Fig. 1 (cf. [30], [31]). This distribution of reliability for the relevant lifetime then needs to be combined with the beta- or binomial distribution of the SR test of prior knowledge using Equation 9 or Equation 10 respectively.

6. EXEMPLARY CASES

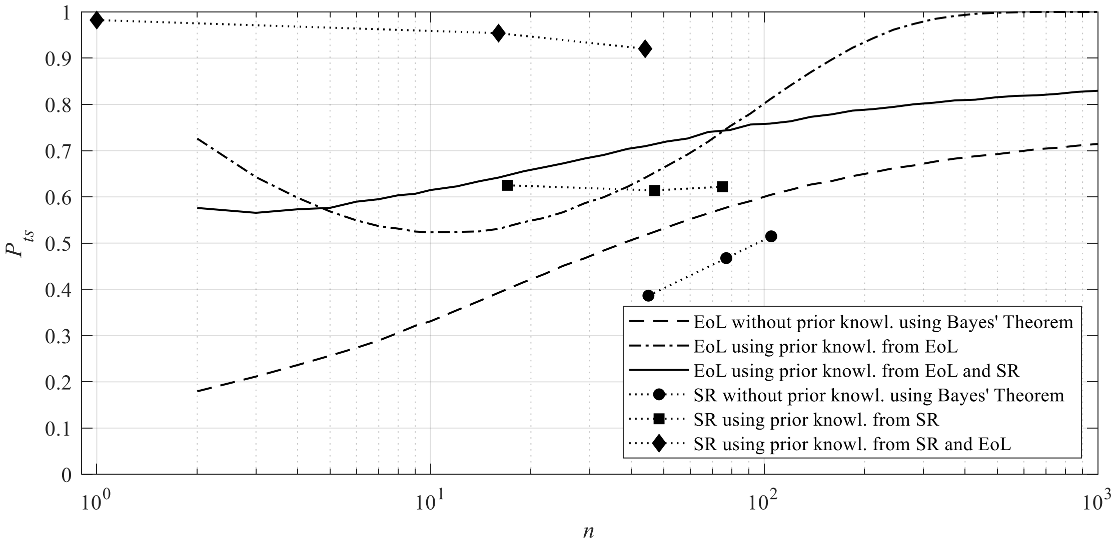
For an analysis of the interaction of the parameters and boundary conditions of the test planning and reliability demonstration using the P_{ts} and Bayes' Theorem via the presented procedures, some exemplary cases are illustrated in the following. Prior knowledge shall be available as listed in table 3 as well as the reliability requirement.

Table 3: Available Prior Knowledge and Reliability Requirement of the Exemplary Cases

<table>
<tr><td>Reliability Requirement:</td><td colspan="6">$R_\mathrm{r} = 0.95$; $C_\mathrm{r} = 0.9$; $t_\mathrm{r} = 0.28$</td></tr>
<tr><td>Prior Knowledge of an SR test</td><td colspan="6">$n_{0,\mathrm{SR}} = 10$ specimen survived the lifetime of $t = 0.4$.
Equivalent to $A_{0,SR} = 28.34$ and $B_{0,\mathrm{SR}} = 1$ for the lifetime $t_\mathrm{r} = 0.28$.</td></tr>
<tr><td rowspan="4">Prior Knowledge of an EoL test</td><td colspan="6">$n_{0,\mathrm{EoL}} = 12$ Weibull distributed failure times with
$T_0 = 0.9274$ and $b_0 = 2.9292$:</td></tr>
<tr><td>0.3933</td><td>0.6369</td><td>0.8355</td><td>0.9122</td><td>0.9004</td><td>0.5688</td></tr>
<tr><td>0.6859</td><td>1.1409</td><td>0.4428</td><td>0.8663</td><td>1.0086</td><td>1.5254</td></tr>
<tr><td colspan="6">Resulting beta distribution for $t_\mathrm{r} = 0.28$: $A_{0,\mathrm{EoL}} = 35.85$, $B_{0,\mathrm{EoL}} = 1.13$</td></tr>
</table>

If an SR test is to be planned, the procedures of section 5.1 are conducted. The resulting values of P_{ts} can be seen in Fig. 2 for $k = 0, 1, 2$ allowed failures. Higher numbers of allowed failures require higher samples sizes overall. For prior knowledge being only used for P_{ts} calculation, the required sample size is $n_\mathrm{r} = 45, 77$ and 105 for the respective number of allowed failures, whereas for the case of considering the prior knowledge of the available SR test, the sample sizes are reduced to $n_\mathrm{r} = 17, 47$ and 75. If both the prior knowledge from the SR test and the EoL test is used, the required sample sizes reduce additionally to $n_\mathrm{r} = 1, 16$ and 44. Using only one specimen in SR testing however, is not advisable.

Figure 2: P_{ts} of the SR Tests and the EoL Tests for Three Scenarios of Considering Prior Knowledge.



It can be seen, that the SR test without consideration of prior knowledge has a P_{ts} which is below 50 % for none and only one allowed failure. For two allowed failures, the P_{ts} is at 51 % but the required sample size is quite high with 105 required specimens. If the prior knowledge from the available SR test is being used, the required sample sizes reduce significantly and therefore also the values in P_{ts} increase all to a moderate level of 61-62 %. The higher estimated reliability due to the consideration of the prior knowledge also results in higher values of P_{ts}. Due to the additionally needed samples if additional failures are allowed during testing, the P_{ts} does not rise significantly. If only the SR test of prior knowledge is considered, it would be unnecessary to allow for failures. If both sources of prior knowledge are considered, the required sample size shrinks an additional significant step. For no allowed failures only one single specimen is required. In addition, the values of P_{ts} again increase to above 90 %. This is due to the lower sample sizes but also due to the additionally considered prior knowledge of the EoL test, which results in attesting a higher probability of survival and therefore a higher probability of the SR test succeeding. As with the case of only considering the prior knowledge from the SR test, the P_{ts} decreases for allowing for more failures. This is due to more specimen having to survive the testing time. In conclusion: the SR test only becomes suitable at all, if the prior knowledge is considered using Bayes' Theorem. This is due to the thereby increased values of P_{ts} and reduced sample sizes. Only by assessing with the P_{ts}, this fact becomes apparent. In general: the expenditure of the SR test can be reduced significantly by using prior knowledge via Bayes' Theorem. The assessment with the P_{ts} enables profound and objective decision making in the planning of reliability demonstrations tests utilizing SR tests. This becomes even more apparent, if the test types are compared against each other.

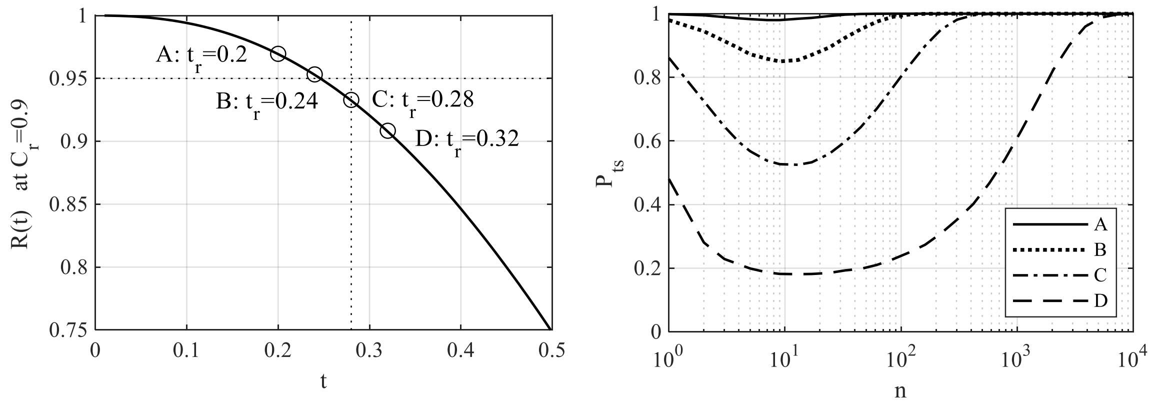
For the planning of an EoL test, several values of the P_{ts} have been calculated for different sample sizes and three scenarios of prior knowledge consideration using Bayes' Theorem: no consideration, only considering the EoL test and considering both the EoL test and the SR test. It can also be seen in Fig. 2. Using no prior knowledge by means of Bayes' Theorem, low values of P_{ts} can be seen. However, moderate to high values are possible if large samples are used. If only the prior knowledge of the EoL test is used with Bayes' Theorem the highest values in P_{ts} can be achieved compared to the other two scenarios analysed here. However, only for sample sizes greater than 80, the advantage is visible. For smaller sample sizes, the consideration of both prior knowledge information bears the advantage of having the highest values in P_{ts}. This is due to the SR test having a greater variance in its reliability distribution than the EoL test, which results in lower values of P_{ts} for higher sample sizes and restrains the P_{ts} in rising to 100 % with very large sample sizes. It can therefore be concluded, that the EoL test greatly benefits from the combined use of P_{ts} assessment and consideration of prior knowledge using

Bayes' Theorem. Here, a sample size of 10-30 is sufficient to guarantee a reliability demonstration test with a moderate chance of success with values of P_{ts} of 60-70 %.

If the SR test and the EoL test is compared, the SR test has the advantage of having an even greater chance of success with an even lower expenditure, due to a lower sample size. The SR test with only one specimen is not feasible, since the uncertainty, which inherits a sample size of one should be avoided in testing. For that reason, the SR test with a sample size of 16 and one allowed failure represents the best SR test. If one specimen costs € 300 and the testing time costs € 100 per time unit, the test cost of this SR test amounts to € 5.248, whereas the median cost of the corresponding EoL test with a sample size of $n = 16$ amounts to € 6.122. However, the EoL test with consideration of both prior knowledge sources, has a value of only $P_{\mathrm{ts}} = 64$ % in contrast to the 95 % of the SR test.

The analysed cases here all correspond to the reliability requirement of Table 3. If, however the required lifetime is changed, a very different behaviour of the P_{ts} values could be seen. Here fore are four cases shown in Fig. 3. On the left, the achievable reliability due to the prior knowledge from the EoL test can be seen for the required confidence of 90 % and required lifetimes from 0 to 0.5. The four cases and their values of P_{ts} can be seen on the right of Fig. 3 for A: $t_r = 0.2$, B: $t_r = 0.24$, C: $t_r = 0.28$ and D: $t_r = 0.32$. If the prior knowledge does already meet the reliability requirement, the curve of P_{ts} starts at very high values but decreases, before it rises again. This is due to the fact, that the prior knowledge is static. If it fulfils the requirement, it always does and taking the prior knowledge without or with a very small additional sample of the planned EoL test, the P_{ts} will be nearly 100 %. The greater the additional sample size is, the greater the impact of sample to sample variability and therefore a decrease in P_{ts}. However, for higher sample sizes, the P_{ts} increases again, which is due to the confidence bounds shrinking and therefore the chance of successfully demonstrating the requirement increases. This behaviour can only be seen, if prior knowledge is being used by means of Bayes' Theorem. For cases without this additional consideration, the P_{ts} is typically monotonically increasing with higher sample sizes.

Figure 3: P_{ts} of the EoL Test for Four Scenarios with Different Required Lifetimes (Right) and Achievable Reliability with Confidence of the Prior Knowledge of the Available EoL Test (Left)



7. CONCLUSION

This paper concerns with two approaches in reliability engineering: The Probability of Test Success, which enables a holistic and objective view on the available test types for reliability demonstration and identifies the one, which will be the most promising. It uses available prior knowledge and calculates the statistical power of the reliability test. Bayes' Theorem on the other hand also uses prior knowledge, but it incorporates it into the reliability statement one obtains from tests. The combination of those two approaches has been established for several cases of available prior knowledge. The thereby newly presented approach enables the planning of efficient reliability demonstration tests by assessing them by means of the Probability of Test Success and also reducing test efforts by using the Bayes' Theorem. Exemplary cases have shown some characteristics, which may occur during practical application. Both the EoL test type as well as the SR test type benefit greatly from incorporating prior knowledge in this way. However, the problem of often very low Probability of Test Success of the SR test can be mitigated by the presented approach. A combined use of different types of prior knowledge is possible. But information stemming from an EoL test is always more valuable in this context than information stemming from an SR test.

The procedures, approaches, analyses and methods presented herein all assume full applicability of the prior knowledge. Since it has very great impact on the statistical information which is obtained, it should undergo rigorous examination and validation with regards to aspects other than the sole reliability information, before the presented approaches commence.

References

[1] M. Dazer, D. Brautigam, T. Leopold, and B. Bertsche, “Optimal Planning of Reliability Life Tests Considering Prior Knowledge,” in *2018 Annual Reliability and Maintainability Symposium (RAMS)*, 2018, pp. 1–7.

[2] M. Dazer, M. Stohrer, S. Kemmler, and B. Bertsche, “Planning of reliability life tests within the accuracy, time and cost triangle,” in *2016 IEEE Accelerated Stress Testing & Reliability Conference (ASTR)*, 2016, pp. 1–9.

[3] A. Grundler, M. Dazer, and T. Herzig, “Statistical Power Analysis in Reliability Demonstration Testing: The Probability of Test Success,” *Appl. Sci.*, vol. 12, no. 12, 2022.

[4] M. Li and W. Q. Meeker, “Application of Bayesian methods in reliability data analyses,” *J. Qual. Technol.*, 2014.

[5] A. Kleyner, S. Bhagath, M. Gasparini, J. Robinson, and M. Bender, “Bayesian techniques to reduce the sample size in automotive electronics attribute testing,” *Microelectron. Reliab.*, 1997.

[6] Y. Liu and A. I. Abeyratne, *Practical Applications of Bayesian Reliability*. Hoboken: John Wiley & Sons, 2019.

[7] A. Grundler, M. Dazer, T. Herzig, and B. Bertsche, “Considering Multiple Failure Mechanisms in Optimal Test Design,” in *Proceedings IRF2020: 7th International Conference Integrity-Reliability-Failure*, 2020, pp. 673–682.

[8] A. Grundler, M. Dazer, T. Herzig, and B. Bertsche, “Efficient System Reliability Demonstration Tests Using the Probability of Test Success,” in *Proceedings of the 31st European Safety and Reliability Conference ESREL 2021*, 2021, pp. 1654–1661.

[9] M. Dazer, A. Grundler, T. Herzig, D. Engert, and B. Bertsche, “Effect on Interval Censoring on the Probability of Test Success in Reliability,” in *2021Annual Reliability and Maintainability Symposium (RAMS)*, 2021.

[10] M. Dazer, T. Herzig, A. Grundler, and B. Bertsche, “R-OPTIMA : Optimal Planning of Reliability Tets,” in *Proceedings IRF2020: 7th International Conference Integrity-Reliability-Failure*, 2020, pp. 695–702.

[11] M. Dazer, T. Leopold, and B. Bertsche, “Optimale Lebensdauertestplanung durch Berücksichtigung von Vorkenntnissen aus stochastischen Betriebsfestigkeitssimulationen,” in *VDI Berichte*, 2016, no. 2279, pp. 777–788.

[12] A. Benz, A. Grundler, T. Herzig, M. Dazer, and B. Bertsche, “Reliability Demonstration Test Planning for Field Load Spectra – an Approach for Identifying the Optimal Test Parameters considering Individual Cost and Time Constraints,” in *Annual Reliability and Maintainybility Symposium RAMS*, 2022.

[13] T. Herzig, M. Dazer, A. Grundler, and B. Bertsche, “Cost-And time-effective planning of accelerated reliability demonstration tests-A new approach of comparing the expenditure of success run and end-of-life tests,” in *Proceedings - Annual Reliability and Maintainability Symposium*, 2019, vol. 2019-Janua.

[14] T. Herzig, M. Dazer, A. Grundler, and B. Bertsche, “Integrating Accelerated Life Tests into Optimal Test Planning,” in *Proceedings IRF2020: 7th International Conference Integrity-Reliability-Failure.*, 2020, pp. 665–672.

[15] T. Herzig, “Anforderungsgerechte Produktauslegung durch Planung effizienter beschleunigter Zuverlässigkeitstests,” Universität Stuttgart, 2021.

[16] T. Herzig, M. Dazer, and B. Bertsche, “Zuverlässigkeitsabsicherung ressourcenschonender Produkte durch effiziente Erprobungsplanung,” in *Stuttgarter Symposium für Produktentwicklung*, 2019.

[17] A. Grundler, M. Dazer, and B. Bertsche, “Reliability-Test Planning Considering Multiple Failure Mechanisms and System Levels,” in *Annual Reliability and Maintainability Symposium. 2020*

Proceedings, 2020.

[18] M. Bayes and M. Price, “An Essay Towards Solving a Problem in the Doctrine of Chances. By the Late Rev. Mr. Bayes, F. R. S. Communicated by Mr. Price, in a Letter to John Canton, A. M. F. R. S.,” *Philos. Trans. R. Soc. London*, vol. 53, pp. 370–418, 1763.

[19] H. F. Martz and R. A. Waller, *Bayesian Reliability Analysis*. New York, Chichester, Brisbane, Toronto, Singapore: John Wiley & Sons, 1982.

[20] A. Kleyner, D. Elmore, and B. Boukai, “A Bayesian Approach to Determine Test Sample Size Requirements for Reliability Demonstration Retesting after Product Design Change,” *Qual. Eng.*, vol. 27, no. 3, pp. 289–295, 2015.

[21] A. Kleyner, S. Bhagath, M. Gasparini, J. Robinson, and M. Bender, “Bayesian techniques to reduce the sample size in automotive electronics attribute testing,” *Microelectron. Reliab.*, vol. 37, no. 6, pp. 879–883, 1997.

[22] A. Krolo, B. Rzepka, and B. Bertsche, “Application of Bayes statistics to reduce sample-size, considering a lifetime-ratio,” *Annu. Reliab. Maintainab. Symp. 2002 Proc. (Cat. No.02CH37318)*, vol. 00, pp. 577–583, 2002.

[23] A. Krolo, M. Components, and S. Stuttgart, “An Approach for the Advanced Planning of a Reliability Demonstration Test based on a Bayes Procedure,” pp. 1–7.

[24] A. Krolo, B. Rzepka, and B. Bertsche, “The Use of the Bayes Theorem to Accelerated Life Tests.”

[25] A. Krolo, “Planung von Zuverlässigkeitstests mit weitreichender Berücksichtigung von Vorkenntnissen,” Universität Stuttgart, 2004.

[26] R. Beyer and E. Lauster, “Statistische Lebensdauerprüfpläne bei Berücksichtigung von Vorkenntnissen,” *Qualität und Zuverlässigkeit, 35, Heft 2*, pp. 93–98, 1990.

[27] A. Grundler, M. Bartholdt, and B. Bertsche, “Statistical test planning using prior knowledge—advancing the approach of Beyer and Lauster,” in *Safety and Reliability – Safe Societies in a Changing World. Proceedings of ESREL 2018*, 2018, vol. 0, pp. 809–814.

[28] M. Bartholdt, A. Grundler, M. Bollmann, and B. Bertsche, “Assurance of the System Reliability of a Gearbox Considering Prior Knowledge,” *Proc. Int. Des. Conf. Des.*, vol. 3, no. 1988, pp. 965–974, 2018.

[29] M. Maisch, “Zuverlässigkeitsorientiertes Erprobungskonzept für Nutzfahrzeuggetriebe unter Berücksichtigung von Betriebsdaten,” Universität Stuttgart, 2007.

[30] A. Grundler, M. Bollmann, M. Obermayr, and B. Bertsche, “Berücksichtigung von Lebensdauerberechnungen als Vorkenntnis im Zuverlässigkeitsnachweis,” in *VDI-Fachtagung Technische Zuverlässigkeit 2019*, 2019.

[31] A. Grundler, M. Göldenboth, F. Stoffers, M. Dazer, and B. Bertsche, “Effiziente Zuverlässigkeitsabsicherung durch Berücksichtigung von Simulationsergebnissen am Beispiel einer Hochvolt-Batterie,” in *30. VDI-Fachtagung Technische Zuverlässigkeit 2021*, 2021.

[32] A. Albers, N. Reiss, N. Bursac, and T. Richter, “IPeM-Integrated Product Engineering Model in Context of Product Generation Engineering,” in *Procedia CIRP*, 2016, vol. 50, pp. 100–105.

[33] A. Albers, N. Bursac, and S. Rapp, “PGE – Produktgenerationsentwicklung am Beispiel des Zweimassenschwungrads,” *Forsch. im Ingenieurwesen/Engineering Res.*, vol. 81, no. 1, pp. 13–31, 2017.

[34] A. Albers, J. Fahl, T. Hirschter, M. Endl, R. Ewert, and S. Rapp, “Model of PGE-Product Generation Engineering by the Example of Autonomous Driving,” in *Procedia CIRP*, 2020, vol. 91, pp. 665–677.

[35] A. Albers *et al.*, “PROPOSING A GENERALIZED DESCRIPTION of VARIATIONS in DIFFERENT TYPES of SYSTEMS by the MODEL of PGE - PRODUCT GENERATION ENGINEERING,” in *Proceedings of the Design Society: DESIGN Conference*, 2020, vol. 1, pp. 2235–2244.

[36] T. Hitziger and B. Bertsche, “An approach to determine uncertainties of prior information - The transformation factor,” *Adv. Saf. Reliab. - Proc. Eur. Saf. Reliab. Conf. ESREL 2005*, vol. 1, pp. 843–849, 2005.

[37] T. Hitziger, B. Bertsche, and A. Krolo, “An Advanced Reliability Test Procedure for Gear-Wheels Considering Results Known from Different Gear Transmission Ratios,” *Probabilistic Saf. Assess. Manag.*, pp. 1894–1899, 2013.

[38] A. K. Gupta and S. Nadarajah, *Handbook of Beta Distribution and Its Applications*. Boca Raton: CRC Press, Taylor & Francis Group, 2018.

[39] R. B. Abernethy, *The new Weibull handbook*. 2006.

[40] M. Dazer, “Zuverlässigkeitstestplanung mit Berücksichtigung von Vorwissen aus stochastischen Lebensdauerberechnungen,” Universität Stuttgart, 2019.

[41] A. Grundler, M. Dazer, and B. Bertsche, “Effect of Uncertainty in Prior Kowledge on Test Planning for a Brake Caliper using the Probability of Test Success,” in *Proceedings - Annual Reliability and Maintainability Symposium*, 2021.

[42] T. Herzig, M. Dazer, A. Grundler, and B. Bertsche, “Cost- and Time-Effective Planning of Accelerated Reliability Demonstration Tests - A New Approach of Comparing the Expenditure of Success Run and End-of-Life Tests,” in *Proceedings - Annual Reliability and Maintainability Symposium*, 2019.
[43] J. Cohen, *Statistical Power Analysis for the Behavioral Sciences*, 2nd ed. New York: Lawrence Erlbaum Associates, 1988.
[44] B. Bertsche, *Reliability in Automotive and Mechanical Engineering*. Berlin: Springer, 2008.
[45] E. A. Karatsuba, “Fast Evaluation of Transcendental Functions,” *Probl. Peredachi Inf.*, vol. 27, no. 4, pp. 76–99, 1991.
[46] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, “NIST Handbook of Mathematical Functions,” no. June 2014, 2010.
[47] M. Li and W. Q. Meeker, “Application of Bayesian methods in reliability data analyses,” *J. Qual. Technol.*, vol. 46, no. 1, pp. 1–23, 2014.
[48] M. S. Hamada, A. G. Wilson, C. S. Reese, and H. F. Martz, *Bayesian Reliability*. New York: Springer-Verlag.
[49] L. Held and D. S. Bové, *Applied Statistical Inference: Likelihood and Bayes*, vol. 9783642378. Heidelberg: Springer-Verlag, 2014.
[50] D. Cousineau and S. Hélie, “Improving maximum likelihood estimation using prior probabilities: A tutorial on maximum a posteriori estimation and an examination of the weibull distribution,” *Tutor. Quant. Methods Psychol.*, vol. 9, no. 2, pp. 61–71, 2013.
[51] H. A. David and H. N. Nagaraja, *Order Statistics*, 3rd ed. Hoboken: John Wiley & Sons, 2003.
[52] B. Bertsche and G. Lechner, *Zuverlässigkeit im Fahrzeug- und Maschinenbau*. Springer-Verlag, 2004.
[53] A. Benard and E. C. Bos-Levenbach, “Het uitzetten van waarnemingen op waarschijnlijkheids-papier,” *Stat. Neerl.*, vol. 7, no. 3, pp. 163–173, Sep. 1953.
[54] T. Hitziger, “Übertragbarkeit von Vorkenntnissen bei der Zuverlässigkeitstestplanung,” Universität Stuttgart, 2007.
[55] A. Krolo, B. Rzepka, and B. Bertsche, “Application of Bayes statistics to reduce sample-size, considering a lifetime-ratio,” 2003.