ASSESSMENT OF STOCHASTIC FATIGUE FAILURES BASED ON DETERMINISTIC FUNCTIONS

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To capture the statistical nature of fatigue crack growth, many stochastic models have been proposed. One of the most common models in probabilistic fracture mechanics analysis proposed by Yang and Manning. This model converts the existing deterministic fatigue model to a stochastic one by adding a random factor. In this study, four different deterministic models are proposed for fatigue life prediction models. These models have more accuracy with less required amount of computation in comparing with the existing ones. It is shown that the proposed models could be used in the Yang and manning for better results.
***Keywords**: fatigue failure, stochastic crack growth analysis, probabilistic fracture mechanics, deterministic function, random process.*

I.INTRODUCTION

When a mechanical structure is subjected to a cyclic stress or strain, after enough cycles, a fatigue crack is initiated on a microscopically scale, and if cycle load continues, these micro-crack growths to a macroscopic size, and finally structure fails as the cyclic loads are applied[1]. This type of failures called fatigue failures. Fatigue failure is one of the most common failures in metallic structures.

Since mechanical property of metallic materials is non-homogeneous, structural applied load has basically random nature and some other reasons; fatigue failure is a random phenomenon and, therefore, needs to be studied stochastically.

Along with the development of fracture mechanics in deterministic fatigue analysis, the so-called probabilistic fracture mechanics has recently received a good attention[2]. One of the important issues in the probabilistic fracture mechanics analysis lies in the probabilistic modeling of fatigue crack growth phenomenon[3].

One of the most common models in probabilistic fracture mechanics analysis proposed by Yang and Manning[4].This method converts deterministic fatigue model (such as Paris-Erdogan law, Elber's law or any other law) to a stochastic one by adding a random factor[3].

Yang and Manning's model are verified by Wu and Ni [3] with using of experimental data. The major reason for adopting Yang and Manning's model by Wu and Ni mainly lies in its generality. Deterministic part of this model is very important, i.e. deterministic function must be match very well with the median trend of the experimental data. Wu and Ni showed that Yang and Manning's model could be modified if a suitable deterministic function be employed [3].

Yang and Manning [5] and C.C.Ni [6] are presented power and polynomial function as deterministic function. In this study for first time four other functions are introduced based on fracture mechanics models and curve fitting technics. These functions have more accuracy or less amount of computation than former ones and are verified by Wu and Ni experimental data and Afgrow software simulation.

II. YANG AND MANNING STOCHASTIC MODEL

In Yang and Manning's stochastic fatigue crack growth model deterministic fatigue crack growth have been randomized to be [3]

$$\frac{da(t)}{dt} = X(t) L(\Delta \mathrm{K}, \mathrm{R}, \mathrm{K}_{\max}, S, a) \quad (1)$$

Where L indicates a general non-negative deterministic crack growth rate function that, as mentioned before, can be Paris-Erdogan law, Elber's law or any other law, ΔK is the stress intensity factor range, R is the stress ratio, K_{max} is the maximum stress intensity factor in a load cycle, S is the maximum stress level, and a(t) is the crack length at time t. In the above equation a random factor X(t) is added into the deterministic crack growth equation to take into account the statistical variability of crack growth rate. After extensive study, Yang and Manning suggested that X(t) could better be modeled as a stationary lognormal random process having a mean value of 1.0 and a standard deviation of σ_x. Under this circumstance, the following normal random process can be introduced [3]

$$Z(t) = \ln X(t) \quad (2)$$

which should have a mean value of zero and a standard deviation of

$$\sigma_Z = \sqrt{\ln(1+\sigma^2{}_X)} \quad (3)$$

The median service time for a crack to grow from size a_0 to a can be obtained from

$$\bar{t}(a) = \int_{a_0}^{a} \frac{dv}{L(v)} \quad (4)$$

To take the random part into consideration, the following integration can also be performed

$$\int_{a_0}^{a} \frac{dv}{L(v)} = \int_0^t X(\tau) d\tau \quad (5)$$

If a new random process W(t) is defined as

$$W(t) = \int_0^t X(\tau) d\tau \quad (6)$$

then W(t) can also be assumed appropriately as a lognormal random process. Therefore the following associated normal random variable Y(t) can be defined

$$Y(t) = \ln W(t) \quad (7)$$

which is assumed to have a mean value $\mu_Y(t)$ and standard deviation $\sigma_Y(t)$. Their values are related to the mean value and standard deviation of W(t)[3].

Finally the probability that crack size a(t) will exceed any given crack size a in the service interval (0,t) can be derived as

$$P^e{}_{a(t)}(a) = P[a(t) > a] = 1 - F_{a(t)}(a) = 1 - F_{W(t)}[\bar{t}(a)] = \Phi\{\frac{\mu_Y(t) - \ln[\bar{t}(a)]}{\sigma_Y(t)}\} \quad (8)$$

The above probability is frequently referred to as crack exceedance probability[3]. As seen Yang and Manning's model is needed to define appropriate deterministic function L. Yang and Manning and C.C.Ni proposed power and polynomial functions, respectively that will be discussed in later part.

III.POWER FUNCTION

A simple and well-known deterministic relation predicted fatigue crack growth is a power law indicated by Paris-Erdogan[7]:

$$\frac{da(t)}{dt} = C\,(\Delta \mathrm{K})^m \qquad (9)$$

where C and m are material constants. For infinite plate with center through crack ΔK defined as [8]

$$\Delta K = \Delta\sigma\sqrt{\pi a} \qquad (10)$$

where Δσ is the applied stress range. Therefore, for constant amplitude loading range and by using of equation (10) in (9), Paris-Erdogan law becomes:

$$\frac{da(t)}{dt} = C\,(\Delta\sigma\sqrt{\pi a})^m = Q_1 a^{b_1} \;,\; Q_1 = C\,(\Delta\sigma\sqrt{\pi})^m \;\text{ and }\; b_1 = \frac{m}{2} \qquad (11)$$

However, when a large amount of data is studied, some discrepancy is observed in crack propagation theory. Walker [9] improved the Paris-Erdogan law by taking into account stress ratio effect as:

$$\frac{da(t)}{dt} = \frac{C_w\,(\Delta \mathrm{K})^{m_w}}{(1-\mathrm{R})^{(1-\gamma)m_w}} \qquad (12)$$

where γ is a material constant. Similarly, for constant amplitude loading range and by using of equation (10) in (12), Walker law becomes:

$$\frac{da(t)}{dt} = \frac{C_w\,(\Delta\sigma\sqrt{\pi a})^{m_w}}{(1-\mathrm{R})^{(1-\gamma)m_w}} = Q_2 a^{b_2} \;,\; Q_2 = \frac{C_w\,(\Delta\sigma\sqrt{\pi})^{m_w}}{(1-\mathrm{R})^{(1-\gamma)m_w}} \;\text{ and }\; b_2 = \frac{m_w}{2} \qquad (13)$$

As seen, the equations (11) and (13) are quite similar and both are power function with general form of L(a) = Q a^b. After investigation of crack propagation in fastener holes of aircrafts under spectrum loading, Yang and Manning have suggested the following simple form [5]

$$\frac{da(t)}{dt} = X\,(t)Q[\mathrm{a(t)}]^b \qquad (14)$$

in which Q and b are constants to be evaluated from the crack growth observation. For this function median service time and normal random process becomes

$$\bar{t}(a) = \int_{a_0}^{a} \frac{dv}{Qv^b} = \frac{a^{1-b} - a_0^{\;1-b}}{Q(1-b)} \qquad (15)$$

$$Z\,(t) = \ln X\,(t) = \ln\frac{da(t)}{dt} - b\,\ln a(t) - \ln\mathrm{Q} \qquad (16)$$

IV.POLYNOMIAL FUNCTION

C.C.Ni [6]'study on reliability of aircraft structures discovered the mismatch of power function with the median trend of the experimental result. Therefore, polynomial function was proposed instead of power function as:

$$\frac{da(t)}{dt} = X\,(t)\{\mathrm{p} + \mathrm{qa(t)} + \mathrm{r[a(t)]}^2\} \qquad (17)$$

In this circumstance median service time and normal random process becomes

$$\bar{t}(a)=\int_{a_0}^{a}\frac{dv}{p+qv+rv^2}=\frac{2}{\sqrt{G}}\tan^{-1}\frac{(2ra-A+q)\sqrt{G}}{Aq+G+2rAa}\quad, G=4pr-q^2 \text{ and } A=2ra_0+q \qquad (18)$$

$$Z(t)=\ln X(t)=\ln\frac{da(t)}{dt}-\ln\{p+qa(t)+r[a(t)]^2\} \qquad (19)$$

Unlike power function, polynomial function is so complicated and needs more amount of computation.

In the next section some new function will be introduced that have more accuracy or less complicity than these two functions. These functions are categorized in two parts:

a) Functions based on classical fracture models

b) Functions based on curve fitting technique

V. FUNCTIONS BASED ON CLASSICAL FRACTURE MODELS

V.A. Rational Function

Neither of Paris-Erdogan nor Walker models could describe macro-crack phase. Forman [10] proposed a model that could described a third stage of fatigue crack growth plot. Forman model followed by:

$$\frac{da(t)}{dt}=\frac{C_f(\Delta K)^{m_f}}{(1-R)K_C-\Delta K} \qquad (20)$$

Where K_c is fracture toughness of the structure material. Similarly, for constant amplitude loading range and by using of equation (10) in (20), Forman law becomes

$$\frac{da(t)}{dt}=\frac{C_f(\Delta\sigma\sqrt{\pi a})^{m_f}}{(1-R)K_C-(\Delta\sigma\sqrt{\pi a})}=\frac{Q_1a^b}{Q_2-Q_3\sqrt{a}}\quad, Q_1=C_f(\Delta\sigma\sqrt{\pi})^{m_f}\quad, Q_2=(1-R)K_C \text{ and } Q_3=\Delta\sigma\sqrt{\pi} \qquad (21)$$

By adding random factor in rational function, new modified Yang and Manning's Model becomes

$$\frac{da(t)}{dt}=X(t)\frac{Q_1a^b}{Q_2-Q_3\sqrt{a}} \qquad (22)$$

Therefore, median service time and normal random process becomes

$$\bar{t}(a)=\int_{a_0}^{a}\frac{Q_2-Q_3\sqrt{v}}{Q_1v^b}dv=\frac{Q_2}{Q_1(1-b)}[a^{1-b}-a_0^{1-b}]-\frac{Q_3}{Q_1(1.5-b)}[a^{1.5-b}-a_0^{1.5-b}] \qquad (23)$$

$$Z(t)=\ln X(t)=\ln\frac{da(t)}{dt}+\ln(Q_2-Q_3\sqrt{a})-\ln(Q_1)-b\ln a \qquad (24)$$

This model is appropriate for instable fatigue fracture (phase III of fatigue crack growth plot). As seen this function is less complicated than polynomial function.

V.B. Global function

Based on classical fracture mechanics stress intensity factor range defined as [8]

$$\Delta K=(\Delta\sigma\sqrt{\pi a})f(a) \qquad (25)$$

Where f(a) is geometric factor. This factor for cracked plate is

$$f\ (a) = \beta_0 + \beta_1 a + \beta_2 a^2 + \beta_3 a^3 + \beta_4 a^4 = \sum_{i=0}^{4} \beta_i a^i \ \text{ and } \ \beta_i = \frac{C_i}{w^i} \qquad (26)$$

in which C_i and w are constants geometric coefficients and plate's width, respectively. For infinite plate with center through crack these coefficients are

$$C_0 = 1\ ,\ C_1 = C_2 = C_3 = C_4 = 0 \ \Rightarrow f\ (a) = 1 \qquad (27)$$

which is led to equation (10).

Neither of power nor rational functions considers this factor. Therefore, the second suggested function in this study is so-called global function that defined as

$$L(a) = L_1(a) \times L_2(a) \qquad (28)$$

Either of power of rational function could be used as $L_1(a)$. $L_2(a)$ is dependence of geometric factor and $L_1(a)$. For example by employing of power function for finite center through cracked plate, $L_2(a)$ becomes

$$L_2(a) = [\mathrm{f(a)}]^{2b} = [1 + 0.128(\frac{a}{w}) - 0.288(\frac{a}{w})^2 + 1.523(\frac{a}{w})^3]^{2b} \qquad (29)$$

For this case median service time and normal random process becomes

$$\bar{t}(a) = \int_{a_0}^{a} \frac{1}{Qv^b} \times \frac{1}{(\sum_{i=0}^{3} \beta_i v^i)^{2b}} dv = \frac{1}{Q}\{\frac{1}{1-b}[\mathrm{a}^{1-b} - \mathrm{a_0}^{1-b}] - \frac{0.256b}{2-b}[\mathrm{a}^{2-b} - \mathrm{a_0}^{2-b}]\} \qquad (30)$$

$$Z\ (t) = \ln X\ (t) = \ln \frac{da(t)}{dt} - \ln Q - b \ln a(t) - 2b \ln \sum_{i=0}^{3} \beta_i a^i \qquad (31)$$

It is necessary to mentioned for simplification in computation, Taylor's explanation is used to calculate integral (30).

As seen this function is more complicated than former ones; instead it has high level of accuracy and is more appropriate for complicated geometries.

VI. FUNCTIONS BASED ON CURVE FITTING TECHNIQUE

VI.A. The function in form of $y = \dfrac{1}{a_1 e^{b_1 x} + a_2 e^{b_2 x}}$

The first function that is introduced in this part has following form

$$\frac{da(t)}{dt} = X\ (t)[\frac{1}{Q_1 e^{ab_1} + Q_2 e^{ab_2}}] \qquad (32)$$

Median service time and normal random process of this function becomes

$$\bar{t}(a) = \int_{a_0}^{a} (Q_1\, \mathrm{e}^{vb_1} + Q_2\, \mathrm{e}^{vb_2}) dv = Q_1 b_1 e^{ab_1} + Q_2 b_2 e^{ab_2} - Q_1 b_1 e^{a_0 b_1} - Q_2 b_2 e^{a_0 b_2} \qquad (33)$$

$$Z\ (t) = \ln X\ (t) = \ln \frac{da(t)}{dt} + \ln(Q_1 e^{ab_1} + Q_2 e^{ab_2}) \qquad (34)$$

VI.B. The function in form of $y = \dfrac{1}{a_1 x^b + a_2}$

The second function is introduced in following form

$$\frac{da(t)}{dt} = X(t)\left[\frac{1}{Q_1 a^b + Q_2}\right] \quad (35)$$

Median service time and normal random process of this function becomes

$$\bar{t}(a) = \int_{a_0}^{a} (Q_1 v^b + Q_2) dv = \frac{Q_1}{1+b}(a^{1+b} - a_0^{1+b}) - Q_2(a - a_0) \quad (36)$$

$$Z(t) = \ln X(t) = \ln\frac{da(t)}{dt} + \ln(Q_1 a^b + Q_2) \quad (37)$$

VI.C. **Validation**

Experimental data of [3] are used for validation of these two functions. Wu and Ni used compact tension (CT) specimens were cut from a 2024-T351 aluminum alloy plate. The dimensions of the specimens were 50.0 mm wide and 12.0 mm thick. The pre-cracking test started at a crack length of 15.0 mm and extended to the length of 18.0 mm. Sinusoidal signals with maximum of 4.5 kN, minimum of 0.9 kN were used as input loads. Results are shown in Table 1.

TABLE 1 Validation of Curve Fitting Based Functions with Experimental Data

Function	R^2	$\bar{t}(a)$ (cycles)	Error[1] (%)
$y = Qa^b$	0.9731	54576	0.4 (Under)
$y = p + qa + ra^2$	0.9805	53523	1.53 (Over)
$y = \frac{1}{Q_1 e^{ab_1} + Q_2 e^{ab_2}}$	0.9485	54175	0.33 (Over)
$y = \frac{1}{Q_1 a^b + Q_2}$	0.9465	54025	0.61 (Over)

As shown in Table 1 these two functions have very low error and can be used as deterministic function in Yang and Manning's model.

VII. RESULTS AND DISCUSSIONS

AFGROW software simulation is used for comparison of presented functions. AFGROW is commercial software that is developed for simulation of aerospace structure's fatigue failures. Specimens used in simulation were rectangular plate with 254 mm wide, 2.54 mm thick and center through cracked. The length of crack was 2.54 mm. These specimens were chosen from aluminum 2024. Applied load had maximum value of 103.42 kN and stress ratio of 0.5. Results are shown in Table 2.

TABLE 2 Comparison of Proposed Functions

Function	R^2	$\bar{t}(a)$ (cycles)	Error (%)
$y = Qa^b$	1.0	376524	1.68 (Under)
$y = p + qa + ra^2$	0.9977	340962	7.92 (Over)
$y = \frac{Q_1 a^b}{Q_2 - Q_3\sqrt{a}}$	0.9993	379000	2.3 (Under)
$y = \frac{1}{Q_1 e^{ab_1} + Q_2 e^{ab_2}}$	0.9998	340392	8.08 (Over)

[1] Under and over refers to less and more prediction that accurate time, respectively.

Function	R^2	$\bar{t}(a)$ (cycles)	Error (%)
$y = \frac{1}{Q_1 a^b + Q_2}$	1.0	370689	0.1 (Under)
$y = Qa^b \times [\sum_{i=0}^{3} \beta_i a^i]^{2b}$	1.0	373969	0.99 (Under)

As seen in Table 2, power function (the first row of Table 2) has good prediction of median service time. However, general function (the last row of Table 2) is improved this prediction by 41%, however it increased amount of computations. The best prediction of median service time is in the 5th row. This function is improved median service time prediction as 94% with almost same amount of computations. The 4th function has similar prediction of median service time to polynomial function, instead it has very low amount of computation.

VIII.CONCLUSION

From experience gained in recent studies it is found that Yang and Manning's stochastic fatigue crack growth model is very general and versatile. To apply this model, one can start to try a simpler fatigue crack growth function. If the model fails to explain the data satisfactorily, he may try to fit median crack growth curve more accurately using another function [3]. In addition to the former presented functions (power and polynomial), in this study four different functions is introduced. These functions have more accuracy or less amount of computation than former ones.

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